Free dilations in continuous-time

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A commuting family of contractive $\mathcal{C}_0$-semigroups $\{T_i(\cdot)\}_{i=1}^{d}$, on a Hilbert space $\mathcal{H}$ admits a simultaneous unitary dilation, if there exists a commuting family of unitary $\mathcal{C}_0$-semigroups $\{U_i(\cdot)\}_{i=1}^{d}$ on a (typically larger) Hilbert space $\mathcal{H}'$ as well as an isometry $r : \mathcal{H} \to \mathcal{H}'$, such that

$$\prod_{i=1}^{d} T_i(t_i) = r^* \Big(\prod_{i=1}^{d} U_i(t_i) \Big) r$$

holds for all $\mathbf{t} \in \mathbb{R}_{\geq 0}^{d}$. For $d \in \{1, 2\}$, the existence of dilations has been fully established in discrete- and continuous-time by Sz.-Nagy, Andô, and Słociński (see [11, Theorems I.4.2 and I.8.1], [1] [7], and [8, Theorem 2], cf. also [6, 9]). For $d \geq 3$, counter-examples were produced in discrete-time by Parrott, Varopoulos, and Kaijser (see [5, §3] and [12, Theorem 1]). And by making use of interpolations, these were recently extended to the continuous-time setting (see [3, Theorem 1.5 and Corollary 1.7 b)]).

This failure can be alleviated, provided we are willing to drop the commutativity assumption: Suppose now that $\{T_i(\cdot)\}_{i \in I}$ is a (not necessarily commuting) family of contractive $\mathcal{C}_0$-semigroups on $\mathcal{H}$ where I is some (discrete) index set. A *free dilation* shall be defined as a family of unitary $\mathcal{C}_0$-semigroups $\{U_i(\cdot)\}_{i \in I}$ on some Hilbert space $\mathcal{H}'$ as well as an isometry $r : \mathcal{H} \to \mathcal{H}'$, such that

$$\prod_{k=1}^{N} T_{i_k}(t_k) = r^* \Big(\prod_{k=1}^{N} U_{i_k}(t_k) \Big) r$$

holds for all $\{i_k\}_{k=1}^{N} \subseteq I$, $\mathbf{t} \in \mathbb{R}_{\geq 0}^{N}$, $N \in \mathbb{N}$.

Following the research in [4], we shall first attempt to motivate the use of non-commutative families, before proceeding with proof/s of the existence of free dilations in the continuous-time setting (cf. Sz.-Nagy [10] and Bożejko [2] for discrete-time proofs) and consider the connection to *free products*. We shall then explore the possibility of dilations in the case that I is replaced by a topological space. This will involve developing a generalised version of the Trotter–Kato theorem. If time permits, we shall explore applications to evolution families and continuously monitored processes.

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