Low-Temperature Physics
Low-Temperature Physics

With 421 Figures
Preface

Science is often a journey to the limits of the feasible and ascertainable. In low-temperature physics this journey strives towards absolute zero. When Louis Cailletet on December 2nd, 1877, realized a major step in terms of the production of low temperatures, namely the first liquefaction of oxygen, he could hardly imagine the wealth of exciting physical phenomena that would be discovered in this field. Despite the anticipation from everyday experience, which generally equates cold with discomfort and stiffening, condensed matter at low temperatures reveals a wide array of fascinating properties. As the most prominent examples let us mention superfluidity and superconductivity, whose attraction is undiminished since their discovery. With every step towards lower temperatures numerous new insights have resulted, which make the traditional subject of low-temperature physics an attractive and modern research topic.

The present book is based on material from lectures that both authors have given several times at the universities of Heidelberg, Bayreuth and Konstanz. It is focused on the discussion of physical phenomena that become most apparent at low temperatures. The book is mainly aimed at students, and provides a compact and comprehensible introduction to various topics of low-temperature physics. Selection and emphasis of the material is subjective and certainly reflects our personal preferences. However, we have tried to give room for as wide a spectrum of topics as possible. The contents are organized in three parts, entitled quantum fluids, solids at low temperatures and principles of refrigeration and thermometry. Quantum fluids, with their diverse and exotic properties, are discussed in the first five chapters of the book. Here, many aspects of the extraordinary liquid, superfluid $^3$He, could only be touched upon since a thorough discussion of this topic is beyond the scope of the book. Chapters six to ten cover aspects of solids at low temperatures. Naturally, superconductivity has been given the largest space here. Atomic tunneling systems, a topic of our own research, has also been discussed in some detail, since these degrees of freedom considerably influence the properties of many solids at low temperatures. The last two chapters of the book are devoted to the common physical principles and methods of low-temperature production and thermometry. In this section, we have intentionally omitted many technical details – which are admittedly often
important for everyday work in the laboratory, but have little to do with the understanding of the underlying physics.

The citations in the text are intended to provide references for the reader to selected important articles and reviews, and to some historically interesting articles. For further studies, problems related to the material discussed are given at the end of each chapter. In addition, some historic anecdotes have been included in the text to introduce some variety.

This book would never have appeared without the help of many colleagues and coworkers. S. Bandler and G. Seidel have read major parts of the manuscript and have given invaluable advice. We are deeply thankful for the enormous time they have committed to identify errors and shortcomings and to make this book much more readable. In addition, selected topics have been read by D. Einzel, A. Fleischmann, R. Kühn, H.v. Löhneysen, D. Vollhardt, M.v. Schickfus, G. Thummes, and V. Mitrovic. Their suggestions have certainly improved the quality of the book and we are most grateful for their help. Special thanks go to R. Weis who skillfully produced all the figures.

Heidelberg, February 2005

C. Enss
S. Hunklinger


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Part I

Quantum Fluids
Evidence for the existence of the rare noble gas helium was first obtained by the French astronomer Janssen in the visible spectrum of solar protuberances during a total eclipse in 1868 in India [1]. Using a spectrometer, he noticed a hitherto unknown yellow line. Shortly afterwards, his observation was confirmed by Lockyer, an English astronomer [2]. He related the new spectral line to a chemical element that had not been seen on Earth before and suggested the name helium for it after helios, the Greek name for sun. On Earth, the helium was discovered in 1895 by Ramsay [3] and independently by Cleve and Langlet [4] in celvite, a rocksand mineral that contains uranium and thorium. Later in the same year, helium was also found in gas evolving from a spring in Bad Wildbad in the Black Forest in Germany by Kayser [5].

Only a few years after the discovery of helium on Earth, a race took place between several competing low-temperature laboratories with the aim of being the first to liquefy this last of the so-called permanent gases. One major difficulty was to obtain sufficient quantities of helium gas. With the help of his brother, who was the director of the Department of Foreign Trade Relations in Amsterdam, Kamerlingh Onnes, the head of the laboratory in Leiden, managed to get a ship load of monazite sand from North Carolina from which he and his coworkers extracted 360 liters of helium gas. He reached the goal of liquefaction of helium on 10 July 1908 [6]. With the achievement of this milestone he established modern low-temperature physics. For many years, the laboratory in Leiden was the leader in this new field.

1.1 Basic Facts

At room temperature, helium is a light inert gas. It is odorless, colorless, and tasteless, and, after hydrogen, the second most abundant element in the universe. Despite its very simple structure, helium exhibits numerous exotic phenomena in condensed form whose theoretical descriptions are rather complex in many cases. We will take a look at some of the unusual characteristics of quantum fluids and quantum solids in the following chapters. To begin, however, let us introduce some general properties of helium in the remaining part of this chapter.
1.1.1 Terrestrial Occurrence

Helium exists in two stable isotopes. $^4\text{He}$ makes up about 5.2 ppm of the Earth’s atmosphere. This trace amount of helium is not gravitationally bound to the Earth and is constantly lost into space. The Earth’s atmospheric helium is replaced by the decay of radioactive elements in the crust of the Earth. Whereas in the early days of helium research, $^4\text{He}$ gas was mainly obtained from minerals, today it is recovered exclusively from natural-gas deposits. The largest sources of helium-rich natural gas are located in the United States. The chemical composition and the helium content of natural gas differs widely, depending on the geological strata. In some natural-gas wells a helium content up to 7% has been found.

The lighter isotope $^3\text{He}$ was discovered and identified in 1933 by Oliphant, Kinsey and Rutherford [7]. At first, however, it was believed that $^3\text{He}$ should not be stable. This misconception was disproved in 1939 by Alvarez and Cornog in cyclotron accelerator experiments [8]. To obtain $^3\text{He}$ in quantities for use in low-temperature physics experiments, it has to be produced artificially via nuclear reactions, because the concentration of $^3\text{He}$ in helium from natural-gas sources is only 0.14 ppm. Therefore, the production of $^3\text{He}$ relies on waste from nuclear reactors or the waste from various constituents of hydrogen bombs. The nuclear reaction is

$$^6\text{Li} + n \rightarrow ^3\text{H} + ^4\text{He} \rightarrow ^3\text{He} + e^- + \bar{\nu}_e .$$

In the first step of this reaction, tritium is produced that has a half-life of 12.5 years. It undergoes beta decay with $^3\text{He}$ as a product. Significant amounts of $^3\text{He}$ were not available before the end of the 1950s. Nevertheless, Sydoriak, Grilly and Hammel managed to liquefy $^3\text{He}$ as early as 1949 and investigated its properties [9].

1.1.2 Basic Atomic and Nuclear Properties

The electronic structure of helium is the simplest many-body system of all atoms. With two electrons completely filling the $K$ shell ($1s^2$) it has a spherical shape and thus no permanent electric dipole moment. Helium has the smallest known atomic polarizability of $\alpha = 0.1232 \text{ cm}^3 \text{ mol}^{-1}$ and only a very weak diamagnetic susceptibility $\chi_d = -1.9 \times 10^{-6} \text{ cm}^3 \text{ mol}^{-1}$. With an atomic radius of only 31 pm it is the smallest atom. It has the highest ionization energy of about 24.6 eV.

The liquid of both isotopes is colorless and since their index of refraction is very close to unity, they are very difficult to see. Because of their nuclear spin $I = 1/2$, $^3\text{He}$ atoms are fermions and obey Fermi–Dirac statistics, whereas $^4\text{He}$ atoms are bosons with nuclear spin $I = 0$. Besides the two stable isotopes $^3\text{He}$ and $^4\text{He}$, there exist two unstable helium isotopes with relatively long half-lives: $^6\text{He}$ ($T_{1/2} = 0.82$ s), and $^8\text{He}$ ($T_{1/2} = 0.12$ s).
1.1.3 Van der Waals Bond

Van der Waals forces act between all atoms. They are based upon the electrical dipole–dipole interaction. At first glance the appearance of this type of force between helium atoms is surprising, because, as mentioned above, they have perfect spherical symmetry and thus no permanent electrical dipole moment. However, one has to take into account that there are always zero-point fluctuations present in the charge distribution, which result in fluctuating dipole moments. The associated electrical fields induce fluctuating dipole moments at neighboring atoms as well and thus lead to a force between the atoms. This force can be described using the Van der Waals potential \( \phi(r) \propto 1/r^6 \), where \( r \) denotes the interatomic distance. In the framework of a quantum-mechanical treatment of the Van der Waals force one can show that this type of interaction always leads to an energy reduction and thus to an attractive force. The strength of this force is given by the polarizability of the interacting atoms. Since the polarizability is small in the case of helium atoms, the binding force is very weak.

By adding a repulsive potential of the form \( \phi(r) \propto 1/r^{12} \) one obtains the well-known Lennard–Jones potential [10]

\[
\phi(r) = 4\varepsilon \left[ \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^{6} \right],
\]

(1.2)

with the characteristic parameters \( \varepsilon \) and \( \sigma \). For both helium isotopes we have \( \varepsilon/k_B = 10.2 \) K and \( \sigma = 2.56 \) Å. The potential energy is obtained by integrating (1.2)

\[
E_{\text{pot}} = \frac{1}{2} \int_0^\infty \phi(r) 4\pi r^2 n(r) \, dr,
\]

(1.3)

taking into account the radial density function \( n(r) \). The difference in the potential energy of liquid and solid helium originates from the difference in \( n(r) \) of the two phases.

In addition to the very weak binding forces between helium atoms, there is a reduction of the binding by the zero-point motion. Although a detailed calculation of the zero-point energy of helium in the liquid phase is rather complex, we can estimate the size of this effect in a simple model in which we assign each atom a certain ‘cage’ volume \( V \). The ground-state energy (zero-point energy) of a particle with mass \( m \) in such a cage is given by

\[
E_0 = \frac{3\hbar^2 \pi^2}{2mV^{2/3}}.
\]

(1.4)

From this result, we can directly see that the zero-point energy for atoms with a small mass like helium is large and that it increases with decreasing molar volume \( V_m \). In Fig. 1.1a the zero-point energy of \(^4\)He is plotted as a function of the molar volume along with the curves for the potential energy of liquid and solid \(^4\)He.
Neglecting the zero-point motion, the solid phase of $^4$He should be stable at a molar volume of approximately $10 \text{ cm}^3 \text{ mol}^{-1}$. However, if we consider the total energy, which is plotted in Fig. 1.1b, we find that the liquid phase is energetically more favorable. In the liquid phase, helium has a molar volume of $28 \text{ cm}^3 \text{ mol}^{-1}$ at $T = 0$. According to (1.4) the zero-point motion of $^3$He leads to an even stronger reduction of the binding energy than for $^4$He. Although the sizes of $^4$He and $^3$He atoms are essentially identical, the molar volume of liquid $^3$He is about $40 \text{ cm}^3 \text{ mol}^{-1}$, and thus much larger than that of liquid $^4$He. We conclude that the interplay of the weak binding force and the large zero-point energy is responsible for the fact that helium is a permanent liquid, which means that it remains liquid under saturated vapor pressure even for $T \to 0$. In addition, this is also why both helium isotopes have the lowest boiling temperatures and critical points known for any substance in nature. Some important parameters of liquid $^3$He and $^4$He are listed in Table 1.1.

**Table 1.1.** Some important material parameters of $^3$He and $^4$He. After [12,13]

<table>
<thead>
<tr>
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<th>$^3$He</th>
<th>$^4$He</th>
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<tbody>
<tr>
<td>boiling temperature at normal pressure $T_b$ (K)</td>
<td>3.19</td>
<td>4.21</td>
</tr>
<tr>
<td>critical temperature $T_c$ (K)</td>
<td>3.32</td>
<td>5.19</td>
</tr>
<tr>
<td>critical pressure $p_c$ (bar)</td>
<td>1.16</td>
<td>2.29</td>
</tr>
<tr>
<td>density for $T \to 0$ $\rho_0$ (g cm$^{-3}$)</td>
<td>0.076</td>
<td>0.145</td>
</tr>
<tr>
<td>density at boiling point $\rho_b$ (g cm$^{-3}$)</td>
<td>0.055</td>
<td>0.125</td>
</tr>
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1.2 Thermodynamic Properties

In the following sections we will briefly consider some basic thermodynamic properties of the two helium isotopes in their liquid form, including a brief discussion of the specific heat. A more detailed analysis of the specific heat of $^4\text{He}$ is presented in Chap. 2, and for $^3\text{He}$ in Chaps. 3 and 4.

1.2.1 Density

During the first liquefaction of helium in 1908, Kamerlingh Onnes already realized that the density of liquid helium is exceptionally small. The values for the densities of $^3\text{He}$ and $^4\text{He}$ at their boiling points are given in Table 1.1. In addition, in 1911 Kamerlingh Onnes made the surprising observation that the density of $^4\text{He}$ exhibits a maximum at about 2 K [14]. Later investigations showed that there is a sharp kink in the temperature dependence of the density at 2.17 K, and that $^4\text{He}$ expands again below that temperature [15].

In Fig. 1.2 the density of liquid $^4\text{He}$ and $^3\text{He}$ is shown as a function of temperature. Measurements of the density of liquid $^3\text{He}$ were carried out in 1949 when $^3\text{He}$ was liquefied for the first time [16]. The density of $^3\text{He}$ did not show a maximum and, as expected, it was much smaller than that of $^4\text{He}$.

![Fig. 1.2. Temperature dependence of the density of liquid $^3\text{He}$ and $^4\text{He}$ [17]](image)

1.2.2 Specific Heat

The first measurements of the specific heat of liquid $^4\text{He}$ were performed by Dana and Kamerlingh Onnes in 1923. They found an abnormal rise of the specific heat around 2 K. In the publication of their results in 1926 they decided to leave out these data points, because they feared that this anomaly
might have been caused by experimental problems [18]. In 1932 Keesom and Clusius investigated the specific heat of liquid $^4$He again and observed a pronounced maximum at about 2.17 K, which they attributed to a phase transition [19].

Since the true nature of the phase transition was unclear for a long time, the two phases were distinguished by naming them helium I and helium II, where helium I denotes the liquid phase above the transition. It was at first believed that helium II represented a crystalline phase under normal pressure. Within this description the fact that it still looked like a fluid was explained in terms of a liquid crystal with flexible planes. This misconception was disproved in 1938 when X-ray diffraction measurements showed undoubtedly that helium II is, in fact, a liquid phase. Surprisingly, it took more than 30 years from the initial observation to the successful explanation of this phase transition. As we will discuss in Sect. 2.3, the nature of the phase transition at 2.17 K can be understood as Bose–Einstein condensation. One of the most intriguing features of helium II is certainly its ability to flow through narrow capillaries without any friction at all. Following the naming of the frictionless transport of electrons in metals as the superconducting state one often refers to helium II as superfluid helium.

Figure 1.3a shows more recent data of the specific heat of liquid $^4$He as a function of temperature. At a temperature of 2.17 K a pronounced maximum occurs. Because of the shape of this curve, which reminds one of the Greek letter $\lambda$, the transition temperature is often referred to as the lambda point. Since the phase transition at the lambda point depends unambiguously on the bosonic character of $^4$He, the occurrence of a similar transition in $^3$He, which carries a nuclear spin $I = 1/2$, was considered to be very unlikely for a long time. Instead, the absence of a superfluid state in $^3$He was seen as important

**Fig. 1.3.** Specific heat of (a) $^4$He [20] and (b) $^3$He [21] in the temperature range where the transition from the normal to the superfluid phase occurs, as a function of temperature.
evidence for the validity of the interpretation of the phase transition in $^4$He as a Bose–Einstein condensation. However, the explanation of the microscopic nature of superconductivity in the framework of BCS theory (see Sect. 10.3) in 1957 changed that viewpoint and intensified the search for a superfluid phase of $^3$He. Finally in 1972, superfluid phases of $^3$He were discovered by Osheroff, Richardson and Lee in nuclear magnetic resonance (NMR) measurements [22]. In contrast to $^4$He, which exhibits just one superfluid phase, $^3$He has three different superfluid phases, depending on temperature, magnetic field, and pressure. Figure 1.3b displays the temperature dependence of the specific heat of liquid $^3$He in the vicinity of one such transition at normal pressure and zero magnetic field. Compared to $^4$He the phase transition in $^3$He occurs at much lower temperatures, namely in the low millikelvin range.

1.2.3 Latent Heat

Dana and Kamerlingh Onnes were also the first to measure the latent heat $L$ of vaporization of $^4$He in 1924 [18]. They observed a minimum of $L$ at the lambda point. The latent heat of $^3$He is much smaller and, as expected, no corresponding minimum occurs around 2 K. Figure 1.4 shows the latent heat of vaporization of $^4$He and $^3$He as a function of temperature.

![Fig. 1.4. Latent heat of vaporization of $^3$He and $^4$He as a function of temperature. The data points for $^4$He are taken from [18, 23]. The solid line has been deduced from the vapor-pressure measurements using the Clausius–Clapeyron equation [24]. In the case of $^3$He the solid line represents the measured temperature dependence of $L$ according to [17]](image)

1.3 Phase Diagrams

The phase diagrams of $^4$He and $^3$He are, in several ways, remarkably different from these of all other known substances. In the following, we will take a brief look at the phase diagrams of the two helium isotopes and point out their distinctive features.
1.3.1 \(^4\)He

The \(p-T\) phase diagram of \(^4\)He is shown in Fig. 1.5. It is most remarkable that helium has no triple point where gas, liquid and solid phase intersect. It remains liquid under normal pressure even at \(T = 0\) as discussed before. Solid helium can only be produced at pressures above 25 bar. Depending on temperature and pressure one finds three different crystalline modifications. In the whole temperature range – but not at all pressures – solid helium with hcp structure exists. In a small pressure range at temperatures between 1.45 K and 1.75 K, \(^4\)He first solidifies into a bcc structure. For pressures exceeding 1 kbar and temperatures above 15 K, \(^4\)He shows a fcc phase (not shown in Fig. 1.5).

As we have seen before, there are two liquid phases of \(^4\)He: helium I and helium II. The transition from helium I to helium II depends on pressure and is shifted towards lower temperatures with increasing pressure. At the melting curve the lambda transition occurs at \(T = 1.9\) K.

At low temperatures (\(T \approx 0.8\) K) the melting curve exhibits a shallow minimum, which is not deep enough to be visible on the scale of Fig. 1.5. Using the Clausius–Clapeyron equation

\[
\frac{\partial p}{\partial T}_{\text{melting curve}} = \frac{S_\ell - S_s}{V_\ell - V_s},
\]

we can draw conclusions about the entropies of liquid and solid \(^4\)He from the melting curve. Here, \(S\) and \(V\) represent entropy and volume per mole, respectively. The indices \(\ell\) and \(s\) denote liquid and solid state, respectively. Since the molar volume of liquid \(^4\)He is always larger than that of solid \(^4\)He, i.e., \(V_\ell - V_s > 0\), we can conclude from \(\partial p/\partial T < 0\) the surprising fact that...
below \( T \approx 0.8 \text{K} \) the entropy of the solid phase is larger than the entropy of the liquid phase. In other words, the disorder in the solid is larger than in the liquid. The entropy of solid and liquid \(^4\text{He}\) is determined by thermal excitations in this temperature range. It turns out that solid \(^4\text{He}\) has a slightly higher phonon heat capacity than liquid \(^4\text{He}\), because of the low transverse sound velocity in solid \(^4\text{He}\). Therefore, at low temperatures the entropy of solid \(^4\text{He}\) is larger than that of liquid \(^4\text{He}\).

1.3.2 \(^3\text{He}\)

The phase diagram of \(^3\text{He}\) (Fig. 1.6) looks qualitatively very similar to that of \(^4\text{He}\), except for the much more pronounced bcc phase. The hcp phase exists in \(^3\text{He}\) only at pressures above 100 bar. As mentioned before, \(^3\text{He}\) also exhibits a transition from a normal fluid to a superfluid phase, or more precisely into three different superfluid phases. We will return to this point in Chap. 4. The transition temperatures are between 1 mK and 3 mK and thus three orders of magnitude lower than in \(^4\text{He}\).

For \( T \to 0 \), \(^3\text{He}\) solidifies at pressures above 33 bar. Between 30 and 100 bar one finds a bcc structure and above about 100 bar an hcp lattice. The bcc phase in solid \(^3\text{He}\) is much more extended than in \(^4\text{He}\). The reason is the higher zero-point energy that favors a smaller packing density. For temperatures above 18 K and for pressures exceeding 1.3 kbar, \(^3\text{He}\) crystallizes into a fcc structure. As in \(^4\text{He}\), the boundary line between liquid and solid \(^3\text{He}\) – the melting curve – shows an anomaly at low temperatures, which we will discuss in the following section.

![Fig. 1.6. Phase diagram of \(^3\text{He}\) [26]](image-url)
Melting Curve

Figure 1.7a shows again the temperature dependence of the melting curve of $^3$He, which exhibits a pronounced minimum at a temperature of about $T = 320 \text{ mK}$. Using the same reasoning as we did for $^4$He we can conclude that the entropy of solid $^3$He is larger than that of liquid $^3$He at temperatures below the minimum.

This amazing phenomenon is not caused by an anomaly of the phonon spectrum, as was the case for $^4$He, because phonons are not the relevant degrees of freedom in $^3$He at such low temperatures. The essential contribution to the entropy comes from nuclear spins. In the liquid, the entropy varies at low temperatures proportional to $T$ as expected for a Fermi gas (see Sect. 3.1). In solid $^3$He the atoms are strongly localized and the Fermi-gas model is not applicable. At high temperatures the orientation of the localized spins is statistical and their contribution to the entropy is $S_s = R \ln 2$, where $R$ is the universal gas constant (see Sect. 11.5). With decreasing temperatures a transition to an antiparallel arrangement of the spins is found. The transition temperature to the antiferromagnetic state is 0.9 mK. At $T = 0.32 \text{ K}$, the temperature at which the minimum occurs, the entropies of liquid and solid $^3$He are equal. The temperature dependence of the entropy of liquid and solid $^3$He is shown in Fig. 1.7b.

As we will see in Sect. 11.5, it is possible to use the melting curve anomaly as a cooling mechanism below 0.32 K, by applying pressure. This mechanism is called Pomeranchuk cooling. With this technique, temperatures down to about 1 mK can be obtained.
Exercises

1.1 Calculate the zero-point energy of a particle in a cube of side length $L$. Determine the zero-point temperature of H$_2$, $^3$He, $^4$He, Ne and Ar under the assumption that the diameter of the atoms is equal to $L$.

1.2 Quantum effects are expected to become important if the wavelength of neighboring atoms noticeably overlap, i.e., if the thermal de Broglie wavelength becomes equal to the interatomic distance. The latter is given approximately by the diameter of the atoms. Calculate the corresponding temperatures at which the above condition is met for the systems mentioned in 1.1.

1.3 The latent heat of liquid helium at $T = 1$ K is $L = 2.156 \times 10^6 \text{ J m}^{-3}$. Calculate the binding energy of a helium atom on the liquid surface.

1.4 Estimate the temperature of the melting-curve minimum for $^3$He.
2 Superfluid $^4$He – Helium II

One of the most striking properties of helium II is its ability to flow through very small capillaries or narrow channels without experiencing any friction at all. This phenomenon was discovered in 1938 independently by Kapitza [31] and by Allen and his coworker Misener [32]. Kapitza named it superfluidity in analogy to superconductivity, which denotes the lossless transport of electrons in superconductors. To explain this discovery, F. London suggested in 1938 that superfluidity is related to the occurrence of an ordered state in momentum space, as would be expected for a Bose–Einstein condensate [33]. Adopting this viewpoint, Tisza postulated in the same year the phenomenological two-fluid model, which nicely describes many properties of helium II [34]. A great success of this model was the prediction of second sound, which was experimentally observed in superfluid helium a few years later. Between 1941 and 1947 Landau published three landmark papers on two-fluid hydrodynamics, in which he explained the phenomenon of superfluidity as a consequence of the excitation spectrum of helium II [35]. Finally, Feynman showed that the excitation spectrum postulated by Landau can be derived within a quantum-mechanical description [36].

In this chapter, we shall discuss the extraordinary properties of superfluid $^4$He starting with some basic experimental observations, which in particular demonstrate the peculiar behavior of this fascinating liquid. Following this we introduce the two-fluid model and discuss whether Bose–Einstein condensation and the collective excitations proposed by Landau can be used to understand the phenomenon of superfluidity, and whether they provide a microscopic foundation for the two-fluid model.

2.1 Experimental Observations

At high temperatures, liquid $^4$He behaves as a dense classical gas, but at the lambda point at $T_\lambda = 2.17$ K, almost all properties of liquid $^4$He change. Even with the naked eye one can see this dramatic transition. In the normal-fluid state $^4$He boils like an ordinary liquid with bubbles rising constantly in the liquid. Below the lambda transition, however, it suddenly becomes deadly quiet and evaporation only takes place at the free surface of the liquid. In the following we present some additional experimental observations of helium II.
2.1.1 Viscosity and Superfluidity

The first indications for the occurrence of superfluidity came from flow measurements through very thin capillaries and narrow slits [31, 32]. Using the Hagen–Poiseuille law

$$\dot{V} = \frac{\pi r^4}{8} \frac{1}{\eta} \frac{\Delta p}{L},$$

one can conclude from measurements of the flow velocity in narrow capillaries that the viscosity of helium II is several orders of magnitude lower than that of helium I. The quantity $L$ denotes the length of the capillary, $r$ the radius, $\Delta p$ the pressure drop along the capillary and $\dot{V}$ the volume rate of helium transported through it. Some measurements that demonstrate the typical variation of flow velocity $v = \dot{V}/(\pi r^2)$ with pressure are shown in Fig. 2.1a. Besides the extremely low viscosity, two other very remarkable observations can be made, namely that the flow velocity is nearly independent of the pressure gradient along the capillary, and that the flow velocity increases with decreasing diameter of the capillary. The temperature dependence of the viscosity deduced from flow measurements through narrow capillaries is shown in Fig. 2.1b. Above the lambda point, the viscosity is nearly temperature independent, but it falls to an undetectably low value for $T < T_\lambda$.

An important question in this context is whether the viscosity becomes extremely small but finite or whether it actually becomes zero below the lambda transition. To answer this question persistent-mass flows have been generated and monitored [37, 38], analogous to persistent-current experiments with superconductors (see Chap. 10). A torus, containing compressed fine powder is filled with liquid helium and set into rotation above the lambda transition.

![Figure 2.1](image.png)

**Fig. 2.1.** (a) Flow velocity of helium II through capillaries with different diameter as a function of the applied pressure [39, 40]. (b) Temperature dependence of the viscosity of liquid helium as determined from flow experiments with thin capillaries.
point. Because of its viscosity, the helium is dragged along with the torus under these conditions. The rotating torus is cooled below $T_\lambda$ and is gently brought to rest. Subsequently, the evolution of the angular velocity of the helium with time is determined. In several experiments this has been achieved by implementing the torus as part of a superfluid gyroscope. From the observation of a constant angular velocity for many hours one can conclude that the viscosity drops at the lambda point by at least eleven orders of magnitude. Within the accuracy of the experiment, this means that helium II is truly flowing without dissipation.

It has been observed, however, that the results of viscosity measurements on helium II, but not on helium I, depend on the measuring method employed. As we will see later, this very peculiar phenomenon can be explained in the framework of the two-fluid model. Before introducing this model in Sect. 2.2, we will take a brief look at the results obtained with two standard techniques for measuring viscosity: the rotary viscosimeter (Fig. 2.2a) and the oscillating-disc method (Fig. 2.2b). In both experiments, the viscosity does not drop instantaneously to zero at the lambda point but remains finite well below the phase transition. For the rotary viscosimeter, the measured viscosity even increases again on cooling below about 1.8 K and substantially exceeds the viscosity of helium I below 1 K. In contrast, $\eta$ drops steadily below $T_\lambda$ with decreasing temperature if measured with an oscillating disc. In addition, in these experiments with oscillating discs, the damping has quite a strong affect on the amplitude of the oscillation, indicating a nonlinear behavior.

Fig. 2.2. Viscosity of liquid helium as a function of temperature measured (a) with a rotary viscosimeter [41, 42] and (b) with an oscillating disc [43]
2.1.2 Beaker Experiments

Further extraordinary observations were made in connection with so-called beaker experiments [44, 45]. Three basic configurations of this experiment are shown schematically in Fig. 2.3. Superfluid helium will flow over the rim into an empty beaker dipped into a bath of helium II until the levels of helium inside and outside the beaker are equal (left). Raising the beaker afterwards causes the helium II to flow back into the bath until equalization of the levels has taken place again (middle). If the beaker is lifted completely out of the bath, helium II will flow over the rim of the beaker and will drop back into the bath until the beaker is empty (right).

![Fig. 2.3. Schematic illustration of different beaker experiments](image)

Figure 2.4 shows the time evolution of the height of the liquid-helium level in a beaker, which was partially immersed in a helium II bath. In addition, the level of the helium bath itself is shown in this plot as a function of time. At the beginning of the experiment the liquid level in the beaker was 13 cm below the liquid level of the helium bath. In the first 45 min helium flows at a constant rate into the beaker, independent of the actual difference in height of the levels. In fact, even a sudden change of the level difference of 100% after 30 min has no recognizable influence on the transfer rate. However, as soon as the level of the bath is lowered below the level of the helium in the beaker (after 45 min), the direction of the transfer reverses and the beaker starts to empty again with the same transfer rate as during the filling procedure.

Figure 2.5a shows a schematic illustration of an experimental setup that was used for a detailed investigation of the equalization of levels. The liquid level in a long thin tube open at the top is monitored with time. In this way, level changes caused by very small transfer rates can be determined with high accuracy. The result of such a measurement is shown in Fig. 2.5b. At the beginning of the experiment the difference in level height decreases linearly with
time, which means that the transfer rate is independent of the level difference, as in the experiments discussed previously. But surprisingly, at the time when the level difference has almost vanished, an undamped oscillation of the helium level in the beaker is observed. In the particular experiment discussed here, the amplitude of this oscillation was about 0.35 mm. The origin of this oscillation is an overshoot of the flow because of the inertia of the flowing helium film every time the levels are equal, leading to a periodic reversal of the flow. Because of superfluidity the oscillation persists nearly undamped over several minutes in the experiment discussed here. This does not always

Fig. 2.4. Level height of a beaker as a function of time. The level was 13 cm below the surface of the surrounding helium bath at the beginning of the experiment [44]

Fig. 2.5. (a) Schematic illustration of a beaker experiment that allows a detailed determination of the helium transfer. The inner diameter of the tube was only 0.58 mm. The copper base of the beaker was incorporated to provide good thermal contact. For clarity, the helium in the beaker is drawn in light grey, although there is no difference between the inner and outer helium. (b) Time evolution of the helium height in the beaker [45]
happen, because, unless special care is taken, temperature gradients between the inside and the outside of the beaker occur, leading to dissipation and thus to a rapid damping of the oscillation.

2.1.3 Thermomechanical Effect

The thermomechanical effect is another unique property of helium II. A schematic illustration of an experimental setup to observe this effect is shown in Fig. 2.6. Two vessels (A and B), both containing helium II are connected via a very thin capillary. Temperature and pressure are equal in both vessels at the beginning of the experiment and thus the helium levels in the two vessels are the same. Increasing the pressure in A results in a flow of helium towards B. Surprisingly, this causes a difference in temperature in the two vessels. The temperature in B decreases somewhat, whereas it increases in A. Equalizing the pressure difference again brings the system back to its starting condition indicating that this is a reversible process. This experiment clearly shows that there is mass flow in helium II associated with the heat flow. However, the fact that the direction of heat flow is actually opposite to the flow of mass is very peculiar.

![Fig. 2.6. Schematic illustration of the principle of the thermomechanical effect](image)

The reversal of the experiment discussed above, namely generation of a pressure difference by heating makes possible the observation of a very attractive phenomenon, the so-called fountain effect (Fig. 2.7). It was first observed by Allen and Jones in 1938 in connection with thermal transport measurements [46]. The fountain effect can be realized by using a flask with a thin neck immersed in helium at $T < T_\lambda$. The lower part of the flask is filled with a fine compressed powder and is open at the bottom. Above the powder tablet an electrical heater is located in the flask. Without heating, the flask fills up with helium until the level of the bath is reached. Heating the helium in the flask results in a fountain of helium ejected from the top of the flask due to the thermomechanical effect. Stationary fountains with heights up to 30 cm have been achieved in this way. Usually, such fountains show turbulent flow. However, under certain conditions (low heater power,
low temperatures, etc.) fountains can be produced exhibiting pure potential flow, like the one shown in Fig. 2.7b.

2.1.4 Heat Transport

Early experiments on heat transport in superfluid $^4$He indicated that the thermal conductivity of helium II is more than five orders of magnitude larger than that of helium I [48,49]. This extremely high thermal conductivity of the superfluid immediately explains the remarkable observation that the boiling of liquid helium stops suddenly when passing the lambda transition. The temperature distribution becomes homogeneous within the liquid and thus evaporation takes place only at the free surface.

Not only is the heat transport of helium II very high, it also has a number of other unusual properties. Figure 2.8 shows that under certain circumstances a pronounced maximum of the heat current density is observed at about 1.8 K. Using capillaries with large diameters one finds, in addition, that the heat-current density $\dot{q}$ rises proportional to $|\text{grad } T|^{1/3}$. This means, that the thermal transport cannot be described by the usual expression $\dot{q} = -\Lambda \text{grad } T$, because the thermal conductivity $\Lambda$ would not be constant but would diverge for small temperature gradients as $\Lambda \propto |\text{grad } T|^{-2/3}$.

Detailed investigations of the heat flux $Q$ of helium II through very thin capillaries have shown that for small temperature differences the heat flux
is indeed proportional to \( \text{grad } T \) as expected. In fact, one finds in helium II a linear relation between heat flow and temperature gradient in a variety of different experiments in which the heat flux does not exceed a certain critical value. The conditions for which this is true is called the \textit{linear regime}. Results of experiments that exhibit this behavior at low heat flux are shown in Fig. 2.9a. Clearly, the heat flow in these experiments depends linearly on the temperature gradient for small values and not too high temperatures.

It is important to note that the presence of a linear regime is \textit{not} only due to the fact that the nonlinear effects seen at high heat fluxes are small. This becomes clear in Fig. 2.9b in which the thermal resistance \( \Delta T/\dot{Q} \) of helium II
in a thin capillary is plotted as a function of the heat current. At small heat currents the thermal resistance is constant, but changes suddenly at a certain value of the heat flux. As we shall discuss later in more detail, the heat transport in helium II is associated with a mass flow for which turbulences in the liquid arise at a certain critical velocity. This, in turn, leads to a sudden increase of the thermal resistance.

2.1.5 Second Sound

Temperature waves, which propagate with a characteristic velocity, are another very remarkable feature of helium II. Since the propagation of such waves is similar to that of ordinary sound this phenomenon was named second sound. The first experimental observation of second sound was made by Peshkov in 1944. In his early experiments he traced the temperature variation associated with propagating second-sound waves. Later, he improved the accuracy of his measurements by generating standing temperature waves. A sketch of his experimental setup for the investigation of standing second-sound waves is shown in Fig. 2.10a.

Using an electrical heater, periodic temperature waves are generated in a resonator with variable length $L$ containing helium II. The temperature distribution in the resonator is monitored with a thermometer that can be moved with respect to the heater position. At resonance, periodic variations of the temperature in the liquid are observed. The result of measurements at two different frequencies is shown in Fig. 2.10b, indicating the presence of standing waves. In this experiment, the velocity of second sound can be

Fig. 2.10. (a) Schematic drawing of the apparatus used by Peshkov for the generation and detection of standing temperature waves in helium II. (b) Temperature of superfluid helium as a function of the thermometer position obtained at two different resonator modes [53]
determined by the simple relation \( v_2 = 2L\nu/n \) for longitudinal resonances. Here, \( \nu \) denotes the heater frequency and \( n \) the number of half-waves in the resonator. With this setup, it is possible to generate temperature waves with frequencies up to 100 kHz. It is remarkable that the velocity of second sound has been found to be independent of the frequency of the heat pulses up to this experimental limit.

### 2.2 Two-Fluid Model

In this section, we will see that the anomalous properties of helium II can be described phenomenologically with the so-called **two-fluid model**. The basic idea of this concept was first suggested in 1938 by Tisza, in order to describe transport phenomena of helium II. According to this model, helium II behaves as if it were a mixture of two completely interpenetrating fluids with different properties, although in reality this is not the case. To avoid any misunderstanding, it must be clearly stated at the outset that the two fluids cannot be physically separated; it is not permissible even to regard some atoms as belonging to the normal fluid and the remainder to the superfluid component, since all \(^4\)He atoms are identical. But accepting these limits of the physical interpretation, many of the phenomena just described can be relatively clearly understood by formally expressing the density of helium II as the sum of a normal-fluid and a superfluid component:

\[
\varrho = \varrho_n + \varrho_s, \tag{2.2}
\]

where \( \varrho, \varrho_n \) and \( \varrho_s \) denote the total, normal-fluid and superfluid densities, respectively. Both \( \varrho_s \) and \( \varrho_n \) depend on temperature, as shown schematically in Fig. 2.11. At absolute zero, helium II consists entirely of the superfluid component (\( \varrho_s = \varrho \) and \( \varrho_n = 0 \)) and at the lambda point it consists entirely of the normal-fluid component (\( \varrho_s = 0 \) and \( \varrho_n = \varrho \)). As we have seen in

![Fig. 2.11. Density of the superfluid and normal-fluid component in helium II as a function of temperature](image)
Sect. 1.2, the total density $\varrho$ is also slightly temperature dependent (see Fig. 1.2). However, in the following description this weak dependence will be neglected. Furthermore, it is assumed that the superfluid component carries no entropy, exhibits no viscous friction and shows no turbulence. The normal-fluid component, in contrast, is assumed to carry the total entropy of the fluid and to exhibit a finite viscosity. The basic assumptions of the two-fluid model are summarized in Table 2.1.

Table 2.1. Basic assumptions of the two-fluid model

<table>
<thead>
<tr>
<th>Density</th>
<th>Viscosity</th>
<th>Entropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>normal-fluid component</td>
<td>$\varrho_n$</td>
<td>$\eta_n = \eta$</td>
</tr>
<tr>
<td>superfluid component</td>
<td>$\varrho_s$</td>
<td>$\eta_s = 0$</td>
</tr>
</tbody>
</table>

We shall see that these simple assumptions lead to a satisfying phenomenological description of many different transport properties of helium II. After introducing the hydrodynamic equations we shall discuss the experimental observations presented in Sect. 2.1 in terms of the two-fluid model.

2.2.1 Two-Fluid Hydrodynamics

In this section, we will look at the basic hydrodynamic equations of the two component fluids. First, we introduce the momentum density $\mathbf{j}$ of mass flow per unit volume

$$\mathbf{j} = \varrho_n \mathbf{v}_n + \varrho_s \mathbf{v}_s.$$  \hfill (2.3)

Here, $\mathbf{v}_n$ and $\mathbf{v}_s$ denote the velocity of the normal and superfluid component, respectively. Mass conservation is expressed by the continuity equation

$$\frac{\partial \varrho}{\partial t} = -\text{div} \mathbf{j}.$$  \hfill (2.4)

Since the viscosity of the normal-fluid component is very low – several orders of magnitude lower than that of water at 300 K – and its influence in most experiments is only a higher-order effect, we shall neglect the normal-fluid viscosity to a first approximation. In this case, helium II is considered as an ideal fluid, which can be described by the Euler equation, the equivalent of Newton’s second law of motion for continua

$$\frac{\partial \mathbf{j}}{\partial t} + \varrho \mathbf{v} \cdot \text{grad} \mathbf{v} \approx -\text{grad} p,$$  \hfill (2.5)

where $p$ denotes the pressure. If the velocities of the two fluids are not too high, to a good approximation we can neglect terms quadratic in the velocities as indicated in (2.5).
point. Because of its viscosity, the helium is dragged along with the torus under these conditions. The rotating torus is cooled below $T_\lambda$ and is gently brought to rest. Subsequently, the evolution of the angular velocity of the helium with time is determined. In several experiments this has been achieved by implementing the torus as part of a superfluid gyroscope. From the observation of a constant angular velocity for many hours one can conclude that the viscosity drops at the lambda point by at least eleven orders of magnitude. Within the accuracy of the experiment, this means that helium II is truly flowing without dissipation.

It has been observed, however, that the results of viscosity measurements on helium II, but not on helium I, depend on the measuring method employed. As we will see later, this very peculiar phenomenon can be explained in the framework of the two-fluid model. Before introducing this model in Sect. 2.2, we will take a brief look at the results obtained with two standard techniques for measuring viscosity: the rotary viscosimeter (Fig. 2.2a) and the oscillating-disc method (Fig. 2.2b). In both experiments, the viscosity does not drop instantaneously to zero at the lambda point but remains finite well below the phase transition. For the rotary viscosimeter, the measured viscosity even increases again on cooling below about 1.8 K and substantially exceeds the viscosity of helium I below 1 K. In contrast, $\eta$ drops steadily below $T_\lambda$ with decreasing temperature if measured with an oscillating disc. In addition, in these experiments with oscillating discs, the damping has quite a strong affect on the amplitude of the oscillation, indicating a nonlinear behavior.

![Fig. 2.2. Viscosity of liquid helium as a function of temperature measured (a) with a rotary viscosimeter [41, 42] and (b) with an oscillating disc [43](image)](image-url)
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Further extraordinary observations were made in connection with so-called beaker experiments [44, 45]. Three basic configurations of this experiment are shown schematically in Fig. 2.3. Superfluid helium will flow over the rim into an empty beaker dipped into a bath of helium II until the levels of helium inside and outside the beaker are equal (left). Raising the beaker afterwards causes the helium II to flow back into the bath until equalization of the levels has taken place again (middle). If the beaker is lifted completely out of the bath, helium II will flow over the rim of the beaker and will drop back into the bath until the beaker is empty (right).

Figure 2.3. Schematic illustration of different beaker experiments

Figure 2.4 shows the time evolution of the height of the liquid-helium level in a beaker, which was partially immersed in a helium II bath. In addition, the level of the helium bath itself is shown in this plot as a function of time. At the beginning of the experiment the liquid level in the beaker was 13 cm below the liquid level of the helium bath. In the first 45 min helium flows at a constant rate into the beaker, independent of the actual difference in height of the levels. In fact, even a sudden change of the level difference of 100% after 30 min has no recognizable influence on the transfer rate. However, as soon as the level of the bath is lowered below the level of the helium in the beaker (after 45 min), the direction of the transfer reverses and the beaker starts to empty again with the same transfer rate as during the filling procedure.

Figure 2.5a shows a schematic illustration of an experimental setup that was used for a detailed investigation of the equalization of levels. The liquid level in a long thin tube open at the top is monitored with time. In this way, level changes caused by very small transfer rates can be determined with high accuracy. The result of such a measurement is shown in Fig. 2.5b. At the beginning of the experiment the difference in level height decreases linearly with
2.1 Experimental Observations

Fig. 2.4. Level height of a beaker as a function of time. The level was 13 cm below the surface of the surrounding helium bath at the beginning of the experiment [44].

Time, which means that the transfer rate is independent of the level difference, as in the experiments discussed previously. But surprisingly, at the time when the level difference has almost vanished, an undamped oscillation of the helium level in the beaker is observed. In the particular experiment discussed here, the amplitude of this oscillation was about 0.35 mm. The origin of this oscillation is an overshoot of the flow because of the inertia of the flowing helium film every time the levels are equal, leading to a periodic reversal of the flow. Because of superfluidity the oscillation persists nearly undamped over several minutes in the experiment discussed here. This does not always

Fig. 2.5. (a) Schematic illustration of a beaker experiment that allows a detailed determination of the helium transfer. The inner diameter of the tube was only 0.58 mm. The copper base of the beaker was incorporated to provide good thermal contact. For clarity, the helium in the beaker is drawn in light grey, although there is no difference between the inner and outer helium. (b) Time evolution of the helium height in the beaker [45].
happen, because, unless special care is taken, temperature gradients between the inside and the outside of the beaker occur, leading to dissipation and thus to a rapid damping of the oscillation.

2.1.3 Thermomechanical Effect

The \textit{thermomechanical effect} is another unique property of helium II. A schematic illustration of an experimental setup to observe this effect is shown in Fig. 2.6. Two vessels (A and B), both containing helium II are connected via a very thin capillary. Temperature and pressure are equal in both vessels at the beginning of the experiment and thus the helium levels in the two vessels are the same. Increasing the pressure in A results in a flow of helium towards B. Surprisingly, this causes a difference in temperature in the two vessels. The temperature in B decreases somewhat, whereas it increases in A. Equalizing the pressure difference again brings the system back to its starting condition indicating that this is a reversible process. This experiment clearly shows that there is mass flow in helium II associated with the heat flow. However, the fact that the direction of heat flow is actually opposite to the flow of mass is very peculiar.

\begin{center}
\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig2_6}
\caption{Schematic illustration of the principle of the thermomechanical effect}
\end{figure}
\end{center}

The reversal of the experiment discussed above, namely generation of a pressure difference by heating makes possible the observation of a very attractive phenomenon, the so-called \textit{fountain effect} (Fig. 2.7). It was first observed by Allen and Jones in 1938 in connection with thermal transport measurements [46]. The fountain effect can be realized by using a flask with a thin neck immersed in helium at $T < T_\lambda$. The lower part of the flask is filled with a fine compressed powder and is open at the bottom. Above the powder tablet an electrical heater is located in the flask. Without heating, the flask fills up with helium until the level of the bath is reached. Heating the helium in the flask results in a fountain of helium ejected from the top of the flask due to the thermomechanical effect. Stationary fountains with heights up to 30 cm have been achieved in this way. Usually, such fountains show turbulent flow. However, under certain conditions (low heater power,
2.2 Two-Fluid Model

2.2.2 Viscosity Measurements

In this section, we shall revisit the viscosity measurements, which have already been presented in Sect. 2.1. Here, we discuss the surprisingly different results in the context of the two-fluid model.

Flow Through Thin Capillaries Due to viscous damping the normal-fluid component is almost completely blocked in thin capillaries \((v_n \approx 0)\). Only the superfluid component is mobile and is observed in such experiments. Since its motion is frictionless, the measured viscosity is zero below the lambda transition.

Rotary Viscosimeter A rotary viscosimeter consists of two hollow cylinders of different size, one rotating inside the other. The viscosity of the liquid between the cylinders is determined via the torque

\[
M_r = \pi \eta \omega d_r^2 d_s^2 / (d_s^2 - d_r^2)
\]

transferred from the rotating inner cylinder with diameter \(d_r\) to the outer stationary cylinder with diameter \(d_s\). Here, \(\omega\) denotes the angular velocity of the rotation. Since the viscosity of the superfluid component is zero, it applies no torque onto the stationary cylinder. Therefore, only the viscosity of the normal-fluid component \(M_r \propto \eta = \eta_n\) is measured in such experiments, which is nonzero even below \(T_\lambda\).

The temperature dependence of \(\eta_n\) is mainly given by the mean free path \(\ell_n\) of the thermal excitations in helium. The increase of the mean free path with decreasing temperature below about 1.8 K can be explained according to the theory of Landau and Khalatnikov, by the reduction of the scattering of thermal excitations. In their model, they assumed a dilute gas of excitations. This assumption is not valid above 1.8 K and therefore the temperature dependence of the normal-fluid viscosity in this temperature range cannot be explained with this theory.

Oscillating-Disc Viscosimeter The viscosity measurements made with this technique are based on the torque

\[
M_d = \pi \sqrt{\rho \eta} \omega^{3/2} r^4 \Theta(\omega)
\]

acting on an oscillating disc with radius \(r\) in the liquid. Here, \(\Theta(\omega) = \Theta_0 \cos(\omega t - \pi/4)\) denotes the angle of deflection and \(\omega\) the angular frequency of the oscillation. The crucial point is that, in this experiment, it is not just the viscosity that is measured, but the product \(\rho \eta\) of density and viscosity.

Since the superfluid component does not contribute below \(T_\lambda\) \((\eta_s = 0)\), the apparent viscosity is given by \(\rho_n \eta_n\). Therefore, the temperature dependence shown in Fig. 2.5 is understandable because the density of the normal-fluid component decreases rapidly below the lambda point. Using the result for \(\eta_n\) from the measurement with rotary viscosimeters, it is possible to draw conclusions about the temperature dependence of \(\rho_n\). The first direct measurement of this quantity is discussed in the next section.
2.2.3 Determination of $\varrho_n/\varrho$

A schematic drawing of the specially designed torsion oscillator that Andronikashvili used in 1948 to determine the normal-fluid density $\varrho_n$ is shown in Fig. 2.12 [55]. The torsion pendulum consisted of 50 equally spaced aluminum discs with a diameter of 3.5 cm and a thickness of only 13 $\mu$m. The discs were mounted on a rod separated from each other by 210 $\mu$m. The rod with the discs was suspended by a torsion fiber forming an oscillator. The lower part of the oscillator was immersed in liquid helium. In the actual experiment, slow oscillations with a typical period of 30 s were excited and the resulting deflection amplitude was detected optically via a mirror that was rigidly attached to the axis of the pendulum.

The total mass of the pendulum bob determines its moment of inertia. The oscillator was driven at the resonant frequency of its torsional mode. This frequency depends on the ratio of the torsion constant and the moment of inertia of the pendulum. The discs of the viscosimeter were constructed from very light material to achieve as large a change as possible in the moment of inertia for a given change of the entrained mass. The spacing $d$ between the discs and the period of oscillation were chosen so that the viscous penetration depth $\delta = \sqrt{2 \eta_n / \varrho_n \omega}$ was larger than the distance between the aluminum discs throughout the entire experiment. Therefore, the complete normal-fluid component $\varrho_n$, but not the superfluid component $\varrho_s$, was dragged with the discs above and below the lambda point.

In this way, the temperature dependence of $\varrho_n$ can be measured directly. The results of the original measurements by Andronikashvili are shown in Fig. 2.13. Above $T_\lambda$, all the fluid is dragged with the discs and the data reflect the temperature dependence of the density of helium I. Below $T_\lambda$,...
2.2 Two-Fluid Model

However, the period of oscillation decreases sharply, indicating that the fluid in the spaces was not completely entrained by the discs. This result confirms the prediction that the superfluid component has no effect on the torsion pendulum. The temperature dependence of $\varrho_n$ below the lambda point can be described by the empirical formula

$$\varrho_n = \varrho_\lambda \left( \frac{T}{T_\lambda} \right)^{5.6},$$

(2.11)

where $\varrho_\lambda$ denotes the density at the lambda point. For comparison, the normal-fluid density $\varrho_n$ as determined by second-sound measurements is also shown in Figure 2.13. As can be seen, the results of the two methods are in very good agreement. In more recent experiments, the ratio $\varrho_s/\varrho$ has been determined over a wider temperature range (see Sect. 2.5).

### 2.2.4 Beaker Experiments

The explanation of the peculiar behavior of helium II in the beaker experiments is straightforward. The geometrical conditions on which we base our considerations are shown in Fig. 2.14. A helium film forms on surfaces above a bath of helium II. The Van der Waals force between helium atoms and the walls plays a crucial role here. The properties of such helium films can be discussed in a simple way in terms of the chemical potential $\mu$. For films in saturated vapor and thermodynamic equilibrium we have

$$\mu_f = \mu_g = \mu_\ell.$$

(2.12)

Here, the indices f, g and $\ell$ refer to film, gas and liquid.

The fact that the film is located on the beaker walls above the liquid surface indicates that the influence of gravity is compensated by Van der Waals forces. Therefore, we can write

$$\mu_\ell = \mu_\ell + \mu_{\text{grav}} + \mu_{\text{vdW}} = \mu_\ell.$$

(2.13)
2.4 Macroscopic Quantum State

*F. London* repeatedly stressed in different publications that the condensate is a quantum state on a macroscopic scale [11]. Later, this viewpoint was extended to the whole superfluid component since it is assumed that the condensate and the superfluid component are closely related. As we will see, the presence of a macroscopic quantum state has consequences for the properties of helium II. For example, it results in a quantization of circulation and it enables phenomena analogous to the Josephson effect in superconductors.

2.4.1 Wave Function of the Superfluid Component

The macroscopic quantum state present in helium II can be described by the wave function

\[ \psi(r) = \psi_0 e^{i\varphi(r)}, \quad (2.67) \]

where the phase \( \varphi(r) \) is a real-valued function of the position. The amplitude \( \psi_0 \) is constant or, under certain conditions, just slightly position dependent. Henceforth, we shall omit the position dependence. The absolute value of the wave function is given by the number of atoms in the superfluid component per unit volume and can be expressed by

\[ \psi^* \psi = |\psi_0|^2 = \frac{\rho_s}{m_4}. \quad (2.68) \]

Here, \( m_4 \) denotes the mass of \( ^4\text{He} \) atoms. The phase of the macroscopic wave is related to the velocity of atoms. The momentum \( p \) of a helium atom in the superfluid component can be described with the Schrödinger equation

\[ -i\hbar \nabla \psi = p \psi. \quad (2.69) \]

Using (2.67) we find \( p = \hbar \nabla \varphi(r) = m_4 v_s \) and thus

\[ v_s = \frac{\hbar}{m_4} \nabla \varphi(r). \quad (2.70) \]

The velocity of the superfluid component therefore determines the phase shift of the wave function. The phase is constant for \( v_s = 0 \), and changes uniformly for \( v_s = \text{const} \). The phase of the wave function is a well-defined quantity within the entire liquid. We can think of particles being ‘rigidly’ connected, though it should be emphasized that this rigid coupling takes place in momentum space and not in real space. This concept can be verified by investigating helium II under rotation. Corresponding experiments will be discussed in the following section.
2.4.2 Helium II Under Rotation – Quantization of Circulation

In the discussion of the two-fluid model we have assumed that in $\varrho_s$ no turbulence occurs and thus we can write $\text{curl} \, \mathbf{v}_s = 0$. Already in 1941, Landau suggested a test of this assumption in experiments with helium II in a rotating vessel [35]. Even before such experiments were conducted Onsager speculated whether the assumption of $\text{curl} \, \mathbf{v}_s = 0$ is generally valid and suspected the occurrence of vortices in rotating helium II [85]. In 1955, Feynman pointed out that the circulation in helium II should be quantized [86]. The first experimental observation of the quantization of circulation was achieved by Vinen in 1961 [87].

First, we consider the normal-fluid component in a rotating vessel. In analogy to the rotation of a rigid body, the velocity of the normal-fluid component is given by $v_n = \omega r$. Here, $\omega$ denotes the angular velocity and $r$ the radial distance from the axis of rotation. The shape of the liquid mirror surface, or in other words the liquid meniscus, is described by the parabola

$$z = \frac{\omega^2}{2g} r^2. \quad (2.71)$$

Now we will turn to the question: What happens with the superfluid component $\varrho_s$ in a rotating vessel? We have assumed in Sect. 2.2 that no turbulence occurs in $\varrho_s$ and therefore that the condition $\text{curl} \, \mathbf{v}_s = 0$ is valid. For a single connected region this statement can be transformed by applying Stokes’ integral theorem

$$\int_A \text{curl} \, \mathbf{v}_s \cdot d\mathbf{f} = \oint_L \mathbf{v}_s \cdot d\mathbf{l} = 0, \quad (2.72)$$

where $A$ is the area enclosed by the contour $L$. From this it follows directly that $\mathbf{v}_s = 0$ and therefore that the superfluid component $\varrho_s$ is at rest. In 1950, experiments with helium II in a rotating vessel were conducted by Osborne, in which he investigated the meniscus [88]. In the case where the superfluid component does not participate in the rotation one expects a reduction of the centrifugal force corresponding to the ratio $\varrho_n/\varrho$. Since gravity acts, of course, on both components, the meniscus of the helium should flatten below the lambda transition. On the contrary, Osborne observed that the meniscus of helium II is identical with that of a classical fluid even well below the lambda point. This means that the superfluid component $\varrho_s$ is not at rest but participates in the rotation, contrary to expectation! As an example, we show in Fig. 2.34 the meniscus curvature $\gamma = d^2z/dr^2$ of helium II under rotation as a function of the angular velocity. Clearly, the data fall onto the classically expected curve (solid line).

To understand this result we first discuss a different experiment involving the rotation of helium II in an annular container. In this case, we have a multiply connected region. The circulation $\kappa$ is given by
\[ \kappa = \oint_L \mathbf{v}_s \cdot d\mathbf{l} = \int_A \text{curl} \mathbf{v}_s \cdot d\mathbf{f} . \] (2.73)

The use of \( \mathbf{v}_s = \hbar \nabla \varphi(r)/m_4 \) from (2.70) results in

\[ \kappa = \frac{\hbar}{m_4} \Delta \varphi_L , \] (2.74)

where \( \Delta \varphi_L \) denotes the phase difference along the integration path \( L \) within the ring. Since the wave function is a unique function, the phase can only differ by integer multiples of \( 2\pi \), i.e., \( \Delta \varphi = 2\pi n \), for a complete cycle. Therefore, we have

\[ \kappa = \frac{\hbar}{m_4} n \quad \text{with} \quad n = 0, 1, 2, 3, \ldots . \] (2.75)

An experimental proof of this quantization was first obtained by Vinen in 1961 [87]. In his experiments, a thin wire (diameter 25 \( \mu \)m, length 5 cm) was placed in the center of a cylindrical vessel filled with helium II. A characteristic transverse vibration of the wire was excited in a constant magnetic field by passing an alternating current through the wire. Without rotation of the surrounding helium, the transverse vibration of the wire can be described by two degenerate oscillations circularly polarized in opposite senses. Under rotation, the degeneracy is lifted by the Magnus force. The frequency difference \( \Delta \nu \) that is now observed, is given by

\[ \Delta \nu = \frac{\varrho_s}{2\pi \mathcal{M}} \kappa , \] (2.76)

where \( \mathcal{M} \) represents the effective mass per length of the wire plus half of the mass of the liquid displaced.

The data shown in Fig. 2.35 are not from the original experiment, but from a more recent investigation similar to that of Vinen. Clearly, quantized values of the circulation are observed. Starting from zero, the rotational velocity was increased slowly, then reduced again and subsequently the direction of
rotation was reversed. During this sequence distinctive hysteretic effects were visible. In such experiments, indications for a quantization up to $n = 4$ have been found (not shown in Fig. 2.35).

**Fig. 2.35.** Circulation $\kappa$ in units of $\hbar/m_4$ as a function of the angular velocity of the rotating cylinder. The arrows indicate the sequence in which the angular velocity was changed [90]

**Vortices with Quantized Circulation**

We have seen that in a multiply connected region the circulation of the superfluid component can be finite and that its magnitude is quantized. Now we come back to the question why the superfluid component seems to participate in the rotation in singly connected regions. The reason is that vortices occur under rotation having a normal-fluid core so that no singly connected region exists within the superfluid component. The occurrence of vortices in helium II therefore provides the explanation for the ‘classical’ meniscus observed in Osborne’s experiments. A schematic illustration of the situation is shown in Fig. 2.36a.

The occurrence of a normal-fluid core in such vortices can be made plausible in the following way. As in a vortex in a classical liquid, the velocity of the superfluid component rises proportional to $1/r$ with decreasing distance $r$ from the center of the vortex. As soon as the critical velocity is exceeded, superfluidity breaks down and a normal-fluid region is formed. The radial dependence of $\rho_s/\rho$ and $v_s$ in the vicinity of a vortex core is shown in Fig. 2.36b. Using classical hydrodynamics and (2.75) one finds for the velocity of the superfluid component:
Fig. 2.36. (a) Schematic illustration of vortices in a rotating vessel containing helium II. (b) Variation of \( v_s \) and \( \varrho_s/\varrho \) as a function of the distance from the vortex center. The normal-fluid vortex core is indicated by the grey shading.

\[
v_s(r) = \frac{\kappa}{2\pi r} = \frac{1.58 \times 10^{-8}}{r} \text{ m/s}.
\]  \( (2.77) \)

We can estimate the diameter \( d_0 \) of the vortex core if we use as a rough approximation in (2.77) for \( v_s \), the critical velocity for roton formation (see Sect. 2.5.3). In this way, we find the very small value of only a few Å for \( T \to 0 \). The radius of the vortex core corresponds to the correlation length – or healing length. This quantity is defined by the length over which the superfluid density falls from its bulk value to zero.

The energy \( E_v \) per unit length of a vortex can be calculated by integrating the kinetic energy per unit volume associated with the rotation of \( \varrho_s \), i.e.,

\[
E_v = \int_{a_0}^{b} \frac{\varrho_s v_s^2}{2} 2\pi r \, dr. \tag{2.78}
\]

Here, \( a_0 \) denotes the radius of the normal-fluid core and \( b \) is given either by the radius \( R \) of the vessel or by half the distance between the vortices. Using \( \kappa = v_s 2\pi r \) we find

\[
E_v = \frac{\varrho_s \kappa^2}{4\pi} \ln \left( \frac{b}{a_0} \right) \propto n^2. \tag{2.79}
\]

Because of the quadratic dependence \( E_v \propto n^2 \), the creation of many vortices with \( n = 1 \) is energetically more favorable than the creation of a smaller number of vortices with correspondingly higher circulation. The angular momentum \( L_v \) per unit length associated with a single vortex is given by

\[
L_v = \int_{0}^{R} \varrho_s r v_s 2\pi r \, dr = \frac{1}{2} \varrho_s \kappa R^2. \tag{2.80}
\]
It can be shown that the critical angular velocity $\omega_c$ for the formation of the first vortex is given by the condition $\omega_c = E_v / L_v$ (see, for example, [91]) resulting in

$$\omega_c = \frac{h}{2\pi m_4 R^2} \ln \left( \frac{R}{a_0} \right). \quad (2.81)$$

For $R = 1$ cm, we find the small critical angular velocity $\omega_c \approx 10^{-3}$ s$^{-1}$.

The presence of vortices can be shown via electrons trapped in the vortex cores. As we will see in the following section, free electrons create bubbles in helium. Because of the circulation and the action of Bernoulli forces the electrons are captured by the vortex cores. By applying an electric field parallel to the vortex lines, the electrons are pulled along the vortices to the surface and the charge accumulating at the surface can be measured with an electrometer. Under the assumption that the electrons are captured at constant rate, the voltage at the electrometer is proportional to the number of vortices present at a given time. If a new vortex is formed upon increasing the angular velocity, the electrometer signal jumps to a higher value. The result of such an experiment is shown in Fig. 2.37. The nature of this data is somewhat analogous to the measurement of quantized magnetic flux lines in type II superconductors (see Sect. 10.1.4). In fact, there are a number of other analogies between these two phenomena. Just as in vortices, the cores of the flux tubes in superconductors are normal conducting. In addition, we find in both cases a regular spatial arrangement, the so-called Abrikosov lattice.

A very impressive direct visualization of the vortex lattice is possible by accelerating the captured electrons with an electric field and having them hit a phosphorescent screen. In Fig. 2.38 the results of such a measurement

![Figure 2.37](image-url)

**Fig. 2.37.** Electrometer signal as a function of angular velocity. The velocity of rotation of the helium II was increased steadily in this experiment [92].
are shown. In this experiment, the helium II was located in a rotating vessel, 2 mm in diameter and 25 cm long. The angular velocities were in the range between 0.3 and 0.8 rad s$^{-1}$. The image was transferred out of the cryostat by optical fibers, amplified and photographed. With increasing angular velocity the number of vortices increases as expected. The regular arrangement of the vortices is evident.

### 2.4.3 Josephson Effect

In 1962, Josephson predicted that if two superconductors are separated by a thin insulator, the superposition of the wave functions of the bulk superconductors causes a current crossing the junction that depends sinusoidally on the phase difference [95] (see also Sect. 10.4.2). This is a direct consequence of the macroscopic nature of the wave function in superconductors. Since the superfluid component of helium II can also be described by a macroscopic wave function one expects analogous phenomena to occur for the mass flow of helium II through a microscopic aperture, providing a so-called weak link between two vessels containing helium II. The size of the aperture has to be of the order of the healing length. Since, at low temperatures, the healing length is extremely small for helium II ($\xi = 1\ldots2\,\text{Å}$), it is experimentally very difficult to achieve this condition. However, since the healing length diverges for $T \to T_\lambda$ (see Sect. 2.6) one can meet the required condition by performing measurements very close to the lambda transition.